UNCERTAINTY IN MEASUREMENT

Introduction

Specifying uncertainty is our way of saying how certain we are of a measured value. This may sound contradictory, but it is not really. For instance, if I told you that my height was exactly 6.0000000000027300001 feet, you would likely not believe me. However, if I told you my height was between 5 feet 11 3/4 inches and 6 feet, you would recognize that as reasonable. One of the most important aspects of measurement is specifying how well a measurement is known. No measurement is exact.

Types of Uncertainty

An estimate of how good you believe your measurement to be is called the **uncertainty** or **error** in the measurement. A density written 2.71 ± 0.05 g/cm^3 indicates that you believe your measurement falls between 2.66 g/cm^3 and 2.76 g/cm^3. The uncertainty is ±0.05 g/cm^3. To estimate uncertainty, you need to understand what can cause it. Sources of uncertainty broadly fall in two categories: random and systematic.

Systematic uncertainties are errors that can be corrected for if known, such as:

- Zero offset. A voltmeter that indicates 0.02 V when the input is 0.00 V, for example, has a zero offset of 0.02 V.
- Friction.
- Ignoring a calibration report. By ignoring this calibration, systematic uncertainties of 0.6 °C to 1.3 °C are introduced.

Thermometer Reading (°C)	True Temperature (°C)
0.0	0.6
10.0	10.7
50.0	51.3

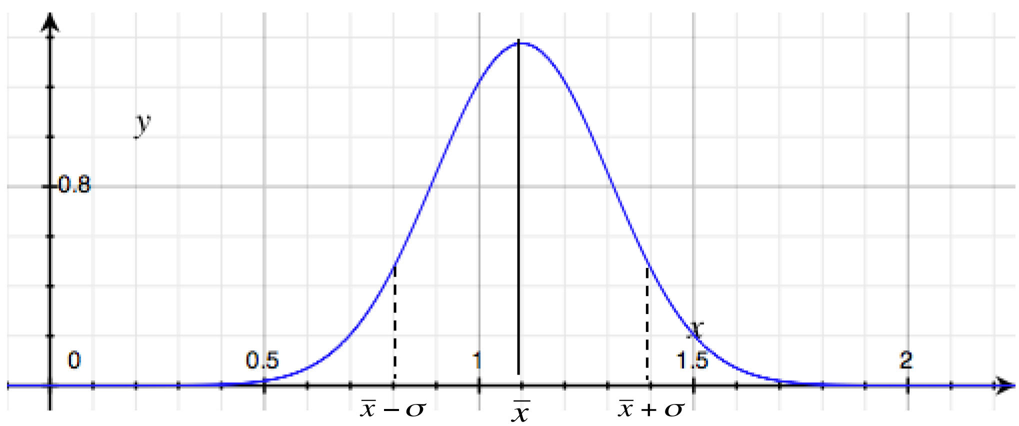
Random uncertainties cannot be corrected for, but can be estimated. Examples are:

- Electrical noise.
- Random fluctuations in temperature, pressure, humidity, etc.
- Variable measurement technique like measuring a diameter in different locations or changing observers.

In taking measurements, the first step is to correct for systematic uncertainties. If not corrected for, they become part of the total uncertainty estimate. Random uncertainties are estimated separately.

Statistical Estimate of Random Uncertainty

Since measurements are not exact, one simple direct way to estimate a value is to repeat measurements. From a set of repeated measurements it is easy to estimate an average value and from the spread in the data we can estimate an uncertainty. If the uncertainty is random and enough repetitions are made, we expect the data to cluster around the average value in what is known as a normal distribution pictured schematically below.



For a data set (x_1, x_2, …, x_n) the average (mean) is usually a good estimate of the true value.

$$\text{Average (Mean)} = \bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum x_i}{n}$$

For the data (1.0, 1.1, 1.2, 1.4, 1.5, 0.8, 0.7)

$$\bar{x} = \frac{1.0 + 1.1 + 1.2 + 1.4 + 1.5 + 0.8 + 0.7}{7} = 1.1$$

If variations in measurements are truly random, then it is just as likely that a given measurement will be above the mean as below. It is a common practice to state the uncertainty in the mean as the estimated standard deviation σ_{est}.

$$\text{estimated standard deviation } = \sigma_{est} = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \bar{x})^2}{N-1}}$$

For the preceding data set,

$$\sigma_{est} = \sqrt{\frac{(1.0-1.1)^2 + (1.1-1.1)^2 + (1.2-1.1)^2 + (1.4-1.1)^2 + (1.5-1.1)^2 + (0.8-1.1)^2 + (0.7-1.1)^2}{6}}$$

$$= 0.29$$

The value represented by the set of measurements would be reported as

$$x = \bar{x} \pm \sigma_{est} \;=\; 1.1 \;\pm\; 0.3$$

which means that roughly 68% of the measurements fell between 0.8 and 1.4.

Reading Errors

Uncertainty arises in individual measurements due to the reading of the instrument used. For an analog scale, the reading error depends on how the instrument is used and the person reading the scale.

The reading at the right edge of the paper is between 4.8 and 4.9 cm. Your eye can probably estimate to one or two tenths of the smallest scale division on this ruler. You might estimate the reading as 4.84 cm. Another person might estimate the reading as 4.83 cm while a third might see the reading as 4.85 cm. Estimating the uncertainty is a bit of a judgment. In this case, about 0.02 cm is reasonable. Thus the reading would be 4.84 ± 0.02 cm.

For a digital readout, the uncertainty is half of the smallest "scale division" or half of the last "digit". The reading on the multimeter pictured is 0.946 V. The smallest division is 0.001 V so the reading is uncertain by 0.0005 V. It would be recorded as 0.9460 ± 0.0005 V.

Accuracy of an Instrument

The above error in reading the multimeter is an indication of the precision of the instrument. However, this is not the whole story. The manufacturer lists the accuracy of this multimeter as ±0.5% of the reading plus one digit. For this measurement then, the accuracy is

$$\pm [(0.946 \text{ X } 0.005) + 0.001] = \pm 0.006$$

The measurement is thus

$$0.946 \pm 0.006 \text{ V}$$

Ordinarily either the error of precision or the accuracy will dominate, and we use the larger as the estimate of uncertainty in the measured value. In this example, the error in the accuracy dominates and is used.

Propagation of Errors

If a physical quantity is related to more than one measurement, what is its uncertainty? For instance, if Z = A + B, what is ΔZ? A maximum uncertainty can be estimated.

$$Z \pm \Delta Z_{\max} = (A \pm \Delta A) + (B \pm \Delta B) = (A + B) \pm (\Delta A + \Delta B)$$
$$\text{i.e., } \Delta Z_{\max} = \Delta A + \Delta B$$

However, this assumes the errors combine in the worst possible way. If A and B are independent measurements, their errors will sometimes add and other times subtract. A common practice is to calculate a Pythagorean sum of the individual errors, also known as adding them in quadrature.

$$\Delta Z = \sqrt{(\Delta A)^2 + (\Delta B)^2}$$

More generally, suppose Z depends on A and B in a more complicated way Z=f(A,B). If A changes by an amount ΔA, then Z changes by $\Delta Z = \frac{\partial f}{\partial A}\Delta A$. If B changes by the amount ΔB, then Z changes by $\Delta Z = \frac{\partial f}{\partial B}\Delta B$. A general expression for the combined error results and can be extended to functions of more than two variables.

Rules for Calculating Errors

Below is a general expression for propagating errors for a function of two variables.

$$\Delta Z = \sqrt{\left(\frac{\partial f}{\partial A}\Delta A\right)^2 + \left(\frac{\partial f}{\partial B}\Delta B\right)^2}$$

This expression results in the following rules for error propagation in simple cases as noted.

If	Then
$Z = A + B \ \ or$ $Z = A - B$	$\Delta Z = \sqrt{(\Delta A)^2 + (\Delta B)^2}$
$Z = A \times B \ or$ $Z = \frac{A}{B}$	$\frac{\Delta Z}{Z} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2}$
$Z = A^n$	$\frac{\Delta Z}{Z} = n\frac{\Delta A}{A}$
$Z = \ln A$	$\Delta Z = \frac{\Delta A}{A}$
$Z = e^A$	$\frac{\Delta Z}{Z} = \Delta A$

