6.1 Plane Waves

Waves will propagate in any medium that has mass and elasticity, or their equivalents in nonmechanical systems. Solid materials, which have both shear and compressive elasticity, allow the propagation of both shear (transverse) and compressive (longitudinal) waves so that their behavior can be very complicated (Morse and Feshbach, 1953, pp. 142–151). Fluids, and in particular gases such as air, have no elastic resistance to shear, though they do have a viscous resistance, and the only waves that can propagate in them are therefore longitudinal, with the local motion of the air being in the same direction as the propagation direction of the wave itself.

When sound waves are generated by a small source, they spread out in all directions in a nearly spherical fashion. We shall look at spherical waves in detail a little later. It is simplest in the first place to look at a small section of wave at a very large distance from the source where the wave fronts can be treated as planes normal to the direction of propagation. In the obvious mathematical idealization, we take these planes to extend to infinity so that the whole problem has only one space coordinate x measuring distance in the direction of propagation.

Referring to Fig. 6.1, suppose that ξ measures the displacement of the air during passage of a sound wave, so that the element ABCD of thickness dx moves to A′B′C′D′. Taking S to be the area normal to x, the volume of this element then becomes

$$V + dV = S\,dx\left(1 + \frac{\partial \xi}{\partial x}\right). \tag{6.1}$$

Now suppose that p_a is the total pressure of the air. Then the bulk modulus K is defined quite generally by the relation

$$dp_{\mathrm{a}} = -K\frac{dV}{V}. \tag{6.2}$$

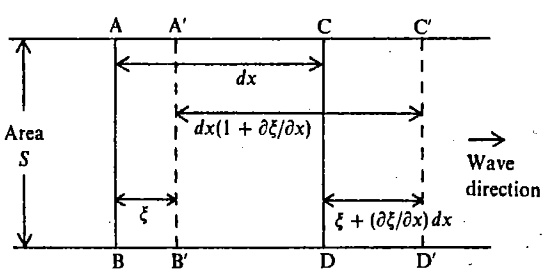

FIGURE 6.1. In passage of a plane wave of displacement ξ, the fluid on plane AB is displaced to A′B′ and that on CD to C′D′.

We can call the small, varying part dp_{a} of p_{a} the sound pressure or acoustic pressure and write it simply as p. Comparison of Eq. (6.2) with Eq. (6.1), noting that V is just $S\,dx$, then gives

$$p = -K\frac{\partial \xi}{\partial x}. \tag{6.3}$$

Finally, we note that the motion of the element ABCD must be described by Newton's equations so that, setting the pressure gradient force in the x direction equal to mass times acceleration,

$$-S\left(\frac{\partial p}{\partial x}\,dx\right) = \rho S\,dx\,\frac{\partial^2 \xi}{\partial t^2},$$

or

$$-\frac{\partial p}{\partial x} = \rho\frac{\partial^2 \xi}{\partial t^2}. \tag{6.4}$$

Then, from Eqs. (6.3) and (6.4),

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{K}{\rho}\frac{\partial^2 \xi}{\partial x^2}, \tag{6.5}$$

or, differentiating Eq. (6.5) again with respect to x and Eq. (6.3) twice with respect to t,

$$\frac{\partial^2 p}{\partial t^2} = \frac{K}{\rho}\frac{\partial^2 p}{\partial x^2}. \tag{6.6}$$

Equations (6.5) and (6.6) are two different versions of the one-dimensional wave equation, one referring to the acoustic displacement ξ and the other to the acoustic pressure p. They apply equally well to any fluid if appropriate values are used for the bulk modulus K and density ρ. For the case of wave propagation in air, we need to decide whether the elastic behavior is isothermal, and thus described by the equation

$$p_{\mathrm{a}}V = \text{constant} = nkT, \tag{6.7}$$

where T is the absolute temperature, or whether it is adiabatic, and so described by

$$p_{\mathrm{a}}V^{\gamma} = \text{constant}, \tag{6.8}$$

where $\gamma = C_p/C_v = 1.4$ is the ratio of the specific heats of air at constant pressure and at constant volume, respectively, and p_{a}, as before, is the average atmospheric pressure.

Clearly, the temperature tends to rise in those parts of the wave where the air is compressed and to fall where it is expanded. The question is, therefore, whether appreciable thermal conduction can take place between these two sets of regions in the short time available as the peaks and troughs of the wave sweep by. It turns out (Fletcher, 1974) that at ordinary acoustic wavelengths the pressure maxima and minima are so far apart that no appreciable conduction takes place, and the behavior is therefore adiabatic. Only at immensely high frequencies does the free-air propagation tend to become isothermal. For sound waves in pipes or close to solid objects, on the other hand, the behavior also becomes isothermal at very low frequencies—below about 0.1 Hz for a 20 mm tube. Neither of these cases need concern us here.

Taking logarithms of Eq. (6.8) and differentiating, we find, using Eq. (6.2),

$$K = \gamma p_a, \tag{6.9}$$

so that Eq. (6.6) becomes

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2}, \tag{6.10}$$

where

$$c^2 = \frac{K}{\rho} = \frac{\gamma p_a}{\rho}, \tag{6.11}$$

and similarly for ξ from Eq. (6.5). As we shall see in a moment, the quantity c is the propagation speed of the sound wave.

It is easy to verify, by differentiation, that possible solutions of the wave equation [Eq. (6.10)] have the form

$$p(x,t) = f_1(x - ct) + f_2(x + ct), \tag{6.12}$$

where f_1 and f_2 are completely general continuous functions of their arguments. We can also see that $f_1(x - ct)$ represents a wave of arbitrary spatial shape $f_1(x - x_0)$ or of arbitrary time behavior $f_1(ct_0 - ct)$ propagating in the $+x$ direction with speed c. Similarly, $f_2(x + ct)$ represents a different wave propagating in the $-x$ direction, also with speed c. In the case of air, or any other nearly ideal gas, Eqs. (6.7) and (6.11) show that

$$c(T) = \left(\frac{T}{T_0}\right)^{1/2} c(T_0), \tag{6.13}$$

where $c(T)$ is the speed of sound at absolute temperature T. There is, however, no variation of c with atmospheric pressure. For air at temperature ΔT degrees Celsius and 50% relative humidity,

$$c \approx 332(1 + 0.00166\,\Delta T)\ \mathrm{m\,s^{-1}}, \tag{6.14}$$

giving $c \approx 343$ m s^{-1} at room temperature.

The wave equation [Eq. (6.10)] was discussed in detail in Chapter 2 in relation to waves on a string, and its two-dimensional counterpart in Chapter 3. There is no need to repeat this discussion here except to remind ourselves that it is usual to treat Eq. (6.10) in the frequency domain where the solutions have the form

$$p = Ae^{-jkx}e^{j\omega t} + Be^{jkx}e^{j\omega t}, \tag{6.15}$$

where $k = \omega/c$ and the A and B terms represent waves traveling to the right and the left, respectively. [If we adopt the conventions of quantum mechanics and write time dependence as $\exp(-i\omega t)$, as for example in Morse (1948), then j should be replaced by $-i$.]

If we consider a wave of angular frequency ω traveling in the $+x$ direction, then we can set $B = 0$ and $A = 1$ in Eq. (6.15) and write

$$p = e^{-jkx}e^{j\omega t} \rightarrow \cos(-kx + \omega t), \tag{6.16}$$

where the second form of writing is just the real part of the first. From Eq. (6.5), ξ has a similar form, though with a different amplitude and perhaps a phase factor. We can connect p and ξ through Eq. (6.4), from which

$$jkp = j\rho\omega\frac{\partial \xi}{\partial t}, \tag{6.17}$$

or, if we write u for the acoustic fluid velocity $\partial\xi/\partial t$ and remember that $k = \omega/c$, then

$$p = \rho c u. \tag{6.18}$$

The acoustic pressure and acoustic fluid velocity (or particle velocity) in the propagation direction are therefore in phase in a plane wave.

This circumstance makes it useful to define a quantity z called the wave impedance (or sometimes the specific acoustic impedance):

$$z = \frac{p}{u} = \rho c. \tag{6.19}$$

$Z_0 = \frac{p}{uS}$ ← in a pipe

It is clearly a property of the medium and its units are Pa m^{-1} s or kg m^{-2} s^{-1}, sometimes given the name rayls (after Lord Rayleigh). For air at temperature ΔT°C and standard pressure,

$$\rho c \approx 428(1 - 0.0017\,\Delta \mathrm{T})\ \mathrm{kg\,m^{-2}s^{-1}}. \tag{6.20}$$

In much of our discussion, we will need to treat waves in 3 space dimensions. The generalization of Eq. (6.10) to this case is

$$\frac{\partial p}{\partial t^2} = c^2\nabla^2 p. \tag{6.21}$$

This differential equation can be separated in several coordinate systems to give simple treatments of wave behavior (Morse and Feshbach, 1953, pp. 499–518, 655–666). Among these are rectangular coordinates, leading simply to three equations for plane waves of the form of Eq. (6.10), and spherical polar coordinates, which we consider later in this chapter.

When a wave encounters any variation in the properties of the medium in which it is propagating, its behavior is disturbed. Gradual changes in the medium extending over many wavelengths lead mostly to a change in the wave speed and propagation direction—the phenomenon of refraction. When the change is more abrupt, as when a sound wave in air strikes a solid object, such as a person or a wall, then the incident wave is generally mostly reflected or scattered and only a small part is transmitted into or through the object. That part of the wave energy transmitted into the object will generally be dissipated by internal losses and multiple reflections unless the object is very thin, like a lightweight wall partition, when it may be reradiated from the opposite surface.

It is worthwhile to examine the behavior of a plane pressure wave Ae^{-jkx} moving from a medium of wave impedance z_1 to one of impedance z_2. In general, we expect there to be a reflected wave Be^{jkx} and a transmitted wave Ce^{-jkx}. The acoustic pressures on either side of the interface must be equal, so that, taking the interface to be at $x = 0$,

$$A + B = C. \tag{6.35}$$

Similarly, the displacement velocities must be the same on either side of the interface, so that, using Eq. (6.19) and noting the sign of k for the various waves,

$$\frac{A - B}{z_1} = \frac{C}{z_2}. \tag{6.36}$$

We can now solve Eqs. (6.35) and (6.36) to find the reflection coefficient:

$$\frac{B}{A} = \frac{z_2 - z_1}{z_2 + z_1}. \tag{6.37}$$

and the transmission coefficient:

$$\frac{C}{A} = \frac{2z_2}{z_2 + z_1}. \tag{6.38}$$

These coefficients refer to pressure amplitudes. If $z_2 = z_1$, then $B = 0$ and $C = A$ as we should expect. If $z_2 > z_1$, then, from Eq. (6.37), the reflected wave is in phase with the incident wave and a pressure maximum is reflected as a maximum. If $z_2 < z_1$, then there is a phase change of 180° between the reflected wave and the incident wave and a pressure maximum is reflected as a minimum. If $z_2 \gg z_1$ or $z_2 \ll z_1$, then reflection is nearly total. The fact that, from Eq. (6.38), the transmitted wave will have a pressure amplitude nearly twice that of the incident wave if $z_2 \gg z_1$ is not a paradox, as we see below, since this wave carries a very small energy.

Perhaps even more illuminating than Eqs. (6.37) and (6.38) are the corresponding coefficients expressed in terms of intensities, using Eq. (6.32). If the incident intensity is $I_0 = A^2/z_1$, then the reflected intensity I_r is given by

$$\frac{I_r}{I_0} = \left(\frac{z_2 - z_1}{z_2 + z_1}\right)^2, \tag{6.39}$$

and the transmitted intensity I_t by

$$\frac{I_t}{I_0} = \frac{4z_2 z_1}{(z_2 + z_1)^2}. \tag{6.40}$$

Clearly, the transmitted intensity is nearly zero if there is a large acoustic mismatch between the two media and either $z_2 \gg z_1$ or $z_2 \ll z_1$.

These results can be generalized to the case of oblique incidence of a plane wave on a plane boundary (Kinsler et al., 1982, pp. 131–133), and we then encounter the phenomenon of refraction, familiar from optics, with the reciprocal of the velocity of sound c_i in each medium taking the place of its optical refractive index.

All these results can be extended in a straightforward way to include cases where the wave impedances z_i are complex quantities $(r_i + jx_i)$ rather than real. In particular, the results [Eqs. (6.39) and (6.40)] carry over directly to this more general situation, the reflection and transmission coefficients generally depending upon the frequency of the wave.

If the surface of the object is flat, on the scale of a sound wavelength, and its extent is large compared with the wavelength, then the familiar rules of geometrical optics are an adequate approximation for the treatment of reflections. It is only for large areas, such as the walls or ceilings of concert halls, that this is of more than qualitative use in understanding behavior (Beranek, 1962; Rossing, 1982; Meyer, 1978).

At the other extreme, an object that is small compared with the wavelength of the sound wave involved will scatter the wave almost equally in all directions, the fractional intensity scattered being proportional to the sixth power of the size of the object. When the size of the object ranges from, for example, one-tenth of a wavelength up to 10 wavelengths, then scattering behavior is very complex, even for simply shaped objects (Morse, 1948, pp. 346–356; Morse and Ingard, 1968, pp. 400–449).

There is similar complexity in the "sound shadows" cast by objects. Objects that are very large compared with the sound wavelength create well-defined shadows, but this situation is rarely encountered in other than architectural acoustics. More usually, objects will be comparable in size to the wavelength involved, and diffraction around the edges into the shadow zone will blur its edges or even eliminate the shadow entirely at distances a few times the diameter of the object. Again, the discussion is complex even for a simple plane edge (Morse and Ingard, 1968, pp. 449–458). For the purposes of this book, a qualitative appreciation of the behavior will be adequate.

Even in an unbounded uniform medium, such as air, a sound wave is attenuated as it propagates, because of losses of various kinds (Kinsler et al., 1982, Chapter 7). Principal among the mechanisms responsible are viscosity, thermal conduction, and energy interchange between molecules with differing external excitation. If we write

$$k \to \frac{\omega}{c} - j\alpha, \tag{6.41}$$

7.6 Radiation from a Plane Source in a Baffle

Referring to Fig. 7.7, suppose that the area S on an otherwise rigid plane baffle is vibrating with a velocity distribution $u(\mathbf{r}')$ and frequency ω normal to the plane, all points being either in phase or in antiphase. The small element of area dS at $\mathbf{r}'$ then constitutes a simple source of volume strength $u(\mathbf{r}')dS$, which is doubled to twice this value by the presence of the plane, which restricts its radiation to the half-space of solid angle 2π. The pressure dp produced by this element at a large distance $\mathbf{r}$ is

$$dp(\mathbf{r}) = \frac{j\omega\rho}{2\pi r} e^{-jk|\mathbf{r}-\mathbf{r}'|} u(\mathbf{r}')dS. \tag{7.28}$$

If we take $\mathbf{r}$ to be in the direction (θ, ϕ) and $\mathbf{r}'$ in the direction $(\pi/2, \phi')$, then we can integrate Eq. (7.28) over the whole surface of the plane, remembering that $u = 0$ outside S, to give

$$p(r, \theta, \phi) = \frac{j\omega\rho}{2\pi r} e^{-jkr} \int\int_S e^{jkr' \sin\theta \cos(\phi-\phi')} u(\mathbf{r}') r'\, d\phi'\, dr'. \tag{7.29}$$

The integral in Eq. (7.29) has the form of a spatial Fourier transform of the velocity distribution $u(\mathbf{r}')$. This is our general result, due in the first place to Lord Rayleigh.

It is now simply a matter of algebra to apply Eq. (7.29) to situations of interest. These include a rigid circular piston and a flexible circular piston (Morse, 1948, pp. 326–335) and both square and circular vibrators excited in patterns with nodal lines (Skudrzyk, 1968, pp. 373–429; Junger and Feit, 1986, Chapter 5). There is not space here to review this work in

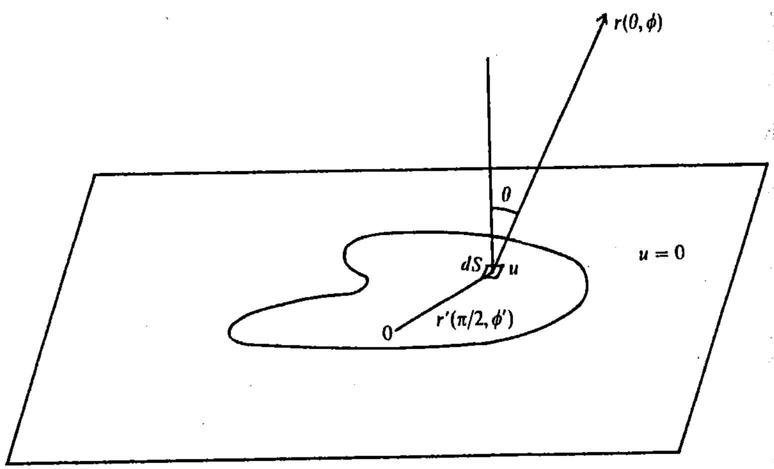

FIGURE 7.7. A vibrating plane source set in an infinite plane baffle. Radiation pressure is evaluated at a point at a large distance r in the direction shown.

detail, but we shall select particular examples and relate the conclusion to the simplified treatments given in the earlier sections of this chapter.

The integral in Eq. (7.29) can be performed quite straightforwardly for the case of a circular piston of radius a with u constant across its surface. The result for the far field (Morse, 1948, pp. 327–328) is

$$p \approx \tfrac{1}{2} j\omega\rho u a^2 \left(\frac{e^{-jkr}}{r}\right) \left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right], \tag{7.30}$$

where J_1 is a Bessel function of order one. The factor in square brackets is nearly unity for all θ if $ka \ll 1$, so the radiation pattern in the half-space $0 \le \theta < \pi/2$ is isotropic at low frequencies. For higher frequencies, the bracket is unity for $\theta = 0$ and falls to zero when the argument of the Bessel function is about 3.83, that is for

$$\theta^* = \sin^{-1}\left(\frac{3.83}{ka}\right). \tag{7.31}$$

The angular width $2\theta^*$ of the primary radiated beam thus decreases nearly linearly with frequency once $ka > 4$. There are some side lobes, but the first of these is already at -18 dB relative to the response for $\theta = 0$, so they are relatively minor.

The force F acting on the piston (Morse, 1948, pp. 332–333; Olson, 1957, pp. 92–93) is

$$F = (R_m + jX_m)u = \rho c S u (A + jB), \tag{7.32}$$

where

$$\begin{aligned} A = 1 - \frac{J_1(2ka)}{ka} &= \frac{(ka)^2}{2} - \frac{(ka)^4}{2^2 \cdot 3} + \frac{(ka)^6}{2^2 \cdot 3^2 \cdot 4} - \cdots \\ &\to \tfrac{1}{2}(ka)^2 \quad \text{for} \quad ka \ll 1; \\ &\to 1 \quad \text{for} \quad ka \gg 1, \end{aligned} \tag{7.33}$$

and

$$\begin{aligned} B = \frac{H_1(2ka)}{ka} &= \frac{1}{\pi k^2 a^2}\left[\frac{(2ka)^3}{3} - \frac{(2ka)^5}{3^2 \cdot 5} + \frac{(2ka)^7}{3^2 \cdot 5^2 \cdot 7} - \cdots\right] \\ &\to 8ka/3\pi \quad \text{for} \quad ka \ll 1, \\ &\to 2/\pi ka \quad \text{for} \quad ka \gg 1, \end{aligned} \tag{7.34}$$

where H_1 is a Struve function of order 1. These functions, which apply also to a pipe with an infinite baffle, are shown later in Fig. 8.7. For the moment, we simply note the close agreement between their asymptotic forms and the same quantities for a pulsating sphere of radius a as given in Eq. (7.25) and Fig. 7.6.