Chapter 1

Introduction

1.1 Dynamical models

To the Greeks, *chaos* signified the infinite formless space which existed before the universe was created. To the generations of thinkers, philosophers and scientists of the succeeding ages, chaos and formlessness have been the subject of countless assaults in an extraordinary search for understanding of the world in which we live.

In the physical sciences these endeavours have been so successful that we can predict the motion of a space craft so as to enable it to be within a few kilometers of a chosen destination after a journey of several years. This level of prediction comes through the use of a *dynamical model*. The model itself is a conceptualisation in which the state of the system is described by state functions which change in time. A *deterministic* dynamical model is one whose future states are uniquely determined from its present state by prescribed laws of evolution. Using such a model, together with powerful computers to solve the equations which encapsulate the laws, makes possible astounding feats of space travel. Yet they do not permit us to predict the weather!

Even our small corner of the universe — the solar system — is too complex to immediately fit a simple model. The process of modelling sorts out the important facts from those of lesser importance, whereupon an account of only the former is undertaken. For example, a model of the solar system which has been the subject of centuries of investigation considers the sun and planets as point masses moving in otherwise empty space, under the sole effect of mutual gravitational forces.

Population models

Dynamical models have been used for the study of populations of species for more than a century. The following quotation will suffice to introduce the idea [1]

[1] David Holton and Robert M. May, "Chaos and one-dimensional maps", in [20], p101.

> In population dynamics, it is desirable to predict trends in populations due to external influences. One of the simplest populations systems is a seasonally breeding organism whose generations do not overlap. ...We seek to understand how the size x_{t+1} of a population in generation $t+1$ is related to the size x_t of the population in the preceding generation. Often an adjustable parameter appears, accounting for, say, the net reproductive rate of the population. We may express such a scalar relationship in general form $x_{t+1} = f(x_t, \lambda)$ ...

The authors go on to discuss some problems with using such a model, but conclude:

> Despite this possibility, there is a rationale for constructing overly simplified models: to capture the essence of observed patterns and processes without being enmeshed in the details.

Consider a simple population model for a single species, in which the reproductive rate[2] is a function $r(x)$ which decreases, with increasing population x, from an initial value $r(0) = r$ to $r(x) = 0$ at some limiting population number K. If we use x to measure the population as a fraction of the carrying capacity K, then the point where $r(x) = 0$ will be at $x = 1$. A simple example is the logistic model, which employs a linear decrease of $r(x)$ with increasing x:

$$r(x) = r(1-x), \qquad f(x) = rx(1-x).$$

Starting from some initial population x_0, this gives rise to the sequence of populations, at successive generations k,

$$x_{k+1} = rx_k(1-x_k).$$

Examples of behaviour

Using CHAOS FOR JAVA[3] one can examine the solution in a number of ways. Figure 1.1 shows the first 50 generations, commencing from an initial population $x_0 = 1/10$, with four different values of the parameter r. Note that if $r < 1$ the population gradually dies out, since the reproductive rate is insufficient for any positive x. Observe the following properties:

[2]This simply aggregates the difference between birth and death rates.

[3]Most of the illustrations in this book were produced with this software, which is free for download. Instructions for obtaining and using it are given in appendix A. You should always attempt to have it at hand when reading, since only a limited number of figures can be printed, conveying only a fraction of what can be learned from your own explorations. In addition, many of the exercises require the use of the software.

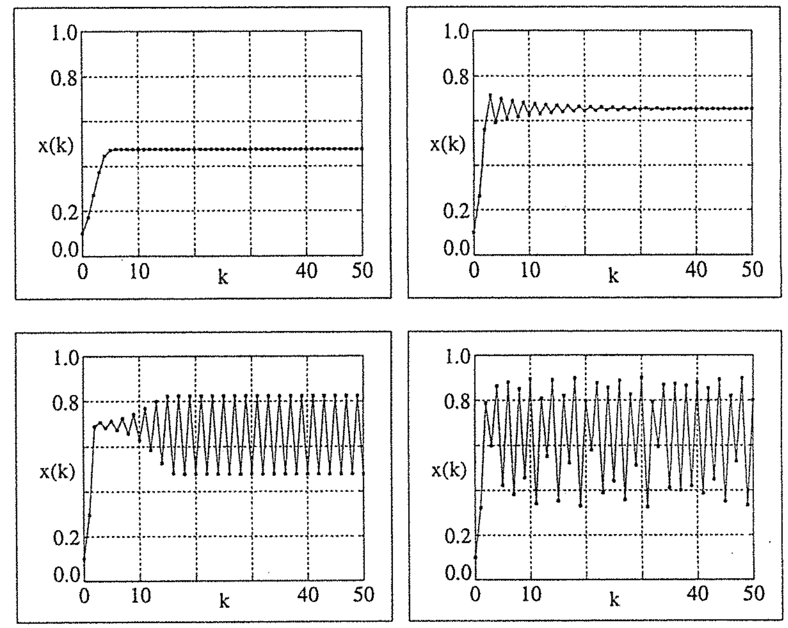

Figure 1.1: Iterations of the logistic map, with parameters (from top left) (i) $r = 1.9$, (ii) $r = 2.9$, (iii) $r = 3.3$, (iv) $r = 3.6$, all with initial value $x_0 = 0.1$.

(i) For $r = 1.9$ the population rises rapidly to a steady value of about 0.47 (47% of the carrying capacity), a figure determined by crowding.

(ii) For $r = 2.9$ the population again stabilises, this time through a sequence of small boom and bust cycles which die out.

(iii) Increasing r to 3.3 changes the behaviour fundamentally. Now the system stabilises on a permanent boom and bust cycle which alternates between good and poor seasons.

(iv) At $r = 3.6$ the behaviour has become extremely complex, with no apparent pattern or simple repetition. It is in fact chaotic.

Imagine the implications for population control policies if such a simple model can generate such disparate outcomes, depending only on the policy settings!

The above figures were produced by the ITERATE(1D) window of CHAOS FOR JAVA with the LOGISTIC MAP selected.[4] Some other interesting values

[4]See appendix A.6 for documentation on the ITERATE(1D) window.

for you to look at before proceeding to the next section include $r = 3.5$, 3.83 and 4.0.

Financial models

Dynamical models are not just the preserve of the sciences. Two professors of economics, for example, introduced a paper with the words [5]

> Imagine a bargaining model ... in which each party has been instructed by higher headquarters to respond to each new offer by her opposite number with a counter-offer that is to be calculated from a simple reaction function ... If the perfectly deterministic sequence of offers and counter-offers that must emerge from these simple rules were to begin to oscillate wildly and apparently at random, the negotiations could easily break down as each party ... came to suspect the other side of duplicity and sabotage. Yet all that may be involved, as we will see, is the phenomenon referred to as chaos ...

Needless to say, there is much research on the question of what non-linear dynamical models and chaos have to say about economics, financial markets [6] and investment management. In this respect it is interesting to note that Benoit Mandelbrot, who coined the word *fractal*, and whose writings had considerable influence in awakening interest in the present subject, first observed the phenomenon of *scaling* in price changes and income distributions. He stated a pricing principle (hypothesis) as follows (see Mandelbrot [17] chapter 37)

> Scaling principle of price change: When $X(t)$ is a price, $\log X(t)$ has the property that its increment over an arbitrary time lag d, $\log X(t+d) - \log X(t)$, has a distribution independent of d, except for a scale factor.

Fractals are treated briefly in chapter 5 of this book; self-similarity of form under changes of scale is one of their hallmarks. An interesting view of how fractals and chaos theory applies to investment theory, including extensive analyses of financial data, may be found in the book of Peters [25].

[5] William J. Baumol and Jess Benhabib, "Chaos: Significance, Mechanism, and Economic Applications", *Journal of Economic Perspectives*, **3**, 77–105 (1989).

[6] In 1838, Thomas Tooke wrote that "the money market turns out always to be in unstable equilibrium", an assertion which has been described as an "absurdity" by modern writers. See page 130 for a typical dynamical system which exhibits exactly such behaviour. For the original quotation, see Blatt [8], p7.

1.2 Celestial mechanics

The earliest dynamical systems which were the subject of intensive study are concerned with Isaac Newton's gravitational model of the solar system. A delightful and easily read account of the history of studies into the solar system is given by Ivars Peterson in his book [26]. I shall give only a brief account here.

Newton's theory gave a satisfactory account of a mass of observations, which had been reduced to three laws by Johannes Kepler. Kepler's laws are

(i) Planetary orbits are plane ellipses with the sun at one focus.

(ii) A line joining a planet with the sun sweeps out area at a rate constant in time.

(iii) The square of the periods of the orbits are proportional to the cubes of the mean radii.

One sees that the very statement of the laws already takes us a long way toward reducing the data to a dynamical model. They define the important state variables as the positions and velocities of the solar bodies, and they state some relationships, although no theoretical explanation is offered.

The triumph of Newton's theory is that these laws are explained as the consequence of a simple dynamical model for which he gives the *equations of evolution*. Newton's second law of motion states that the rate of change of momentum of a body is equal to the sum of the forces acting on it; his gravitational theory states that the force acting between any pair is proportional to the product of their masses, inversely proportional to the square of the distance between them, and directed along the line joining them at any instant of time. The constant of proportionality, G, is a *universal constant* of nature.

Newton was acutely aware of various deficiencies of his theory as applied to the solar system in finer detail. In fact he was unhappy about his inability to give a proper account for the observed motion of the moon. There were other discrepancies too, particularly in the motion of the two largest planets, Jupiter and Saturn. In some brilliant work, Pierre Simon de Laplace accounted for this latter as a mutual near resonant interaction resulting in periodic changes which take approximately 900 years for each cycle. So confident was he of the validity of the underlying methods of dynamics that he wrote [7]

> Assume an intelligence that at a given moment knows all the forces that animate nature as well as the momentary position

[7] Taken from Peterson [26], p229.

> of all things of which the universe consists, and further that it is sufficiently powerful to perform a calculation based on these data. It would then include in the same formulation the motions of the largest bodies in the universe and those of the smallest atoms. To it, nothing would be uncertain. Both future and past would be precise before its eyes.

This is an extreme statement of the view that the solar system — even the entire universe — is a predictable *clockwork* system.

Poincaré and the birth of chaos

In November 1890 Henri Poincaré's memoir on the three-body problem (see below for a brief description) was published as the winning entry in an international competition to honour the 60th birthday of King Oscar II of Sweden and Norway.[8] There were four questions from which the contestants might choose; Poincaré's choice was the one whose solution would, it was hoped, lead to a resolution of the question of the stability of the solar system (or rather, the Newtonian model of the solar system). In part, the question read (Barrow-Green [6], p229)

> A system being given of a number whatever of particles attracting one another mutually according to Newton's law, it is proposed ... to expand the coordinates of each particle in a series ... according to some known functions of time ...

Poincaré's winning entry, and the published memoir (which differed significantly from the original due to the discovery of an important error) were on "the problem of three bodies and the equations of dynamics". In fact, his investigations are concerned with the "restricted circular three body problem". This version has two of the bodies, one rather more massive than the other, in circular orbits about their centre of mass, and seeks to explain the motion of a third body whose mass is too small to influence the two primaries. This simplification of the original question — concerned with an arbitrary number of bodies moving in three dimensions — to three bodies moving in a plane, two of them in fixed circular motion, illustrate the importance of simple models to making progress in fundamental understanding. It underscores the comments of Holton and May, made on population models, and quoted above.

A proper exposition of Poincaré's work requires a substantial volume in itself. Here I just mention a few salient points.

[8]See the book by June Barrow-Green [6] for a rather complete account of Poincaré's contributions to mathematics and dynamics.

(i) Poincaré gave prominence to the geometric properties of the orbits as smooth curves in space, defined by the evolution of the state variables.

(ii) He showed that by making suitable choices in representing the problem, individual orbits may be investigated via the set of points at which they pierce a two-dimensional *transverse surface*.

(iii) The original dynamics is now encoded as the *map* which relates the successive piercings of this surface, since each is determined from the previous one solely by the equations of motion.

(iv) Periodic orbits show up as isolated points in this map.

(v) Entire families of orbits may now be represented as curves in a surface of section, each point on the curve representing an orbit.

(vi) Certain families of orbits lie on curves which intersect themselves infinitely often in the neighbourhood of a single point.

Thus Poincaré instigated a new way to study dynamical systems which emphasised qualitative and geometric features, not just analytical formulae. His method (ii) is widely used today and is known as the method of *Poincaré sections*.[9]

The study of maps, instituted in (iii) is used in the theory of dynamical systems and chaos. The *homoclinic tangles* identified in (vi) play an important part in advanced studies of chaotic systems. Although I shall not be concerned with the theory of such orbits herein, it is important to understand that it was in the process of exploring the *infinite complexity* of such bizarre objects that Poincaré arrived at the doorway to an appreciation of chaos.

Understandably, Poincaré's work, particularly the memoir of 1890, has drawn unceasing admiration for more than a century. 100 years on, one reviewer wrote that the memoir was[10]

> ... the first textbook in the qualitative theory of dynamical systems ...

1.3 Lorenz: the end of weather prediction?

Despite the importance of Poincaré's work, and other work in the first half of the 20th century, the implications for unpredictable and chaotic

[9]In his book "The Essence of Chaos" [16], Edward Lorenz gives a beautifully clear account of the meaning of Poincaré sections without the use of mathematical formulae.
[10]Philip Holmes, "Poincaré, celestial mechanics, dynamical systems theory and chaos", *Physics Reports*, **193**, 137–163 (1990).

behaviour were not widely appreciated until the advent of electronic computation. This is hardly surprising, since the fact that usable analytic formulae cannot be found for relatively uncomplicated dynamical models means that a proper appreciation of the nature of their solutions had to await such a development.

Thus it was not until Edward Lorenz' 1963 paper[11] that a new era opened in non-linear dynamics and chaos. Lorenz considered the relatively harmless looking differential equations

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y-x), \\ \frac{dy}{dt} &= rx - y - xz, \\ \frac{dz}{dt} &= xy - bz. \end{aligned} \tag{1.1}$$

Here x, y and z are the state variables, σ, b and r are parameters which control the types of behaviour (see below for a brief explanation). Were it not for the two non-linear terms (xz in the second equation and xy in the third), the complete set of solutions would be expressible using only the exponential, sine, and cosine functions, and a few constants easily computed from the coefficients b, σ and r, together with the initial values of x, y and z. That is to say, not only would it be a deterministic dynamical system, but more importantly, all possible behaviour patterns would be simple to understand. One facet I want to emphasise here is that, because of the nature of the equations, were they linear then at most one natural frequency would be required for the description of the motion.

Strange attractors

What Lorenz found are solutions which are *nonperiodic*, that is, they cannot be represented using any finite number of frequencies. These solutions are also *sensitively dependent* on initial conditions, which means that for all practical purposes, prediction of the state of the system is limited to relatively short times. Furthermore, in the regime where chaotic solutions exist, then regardless of the initial conditions, they are all attracted to some region of state space whose dimension is not an integer! It resembles a surface with two wings, but it is more like a fat surface, with an infinite number of sheets. Such objects are generally called *strange attractors*, and again we are confronted with the infinite when examining the behaviour of a simple dynamical system.

[11] Edward N. Lorenz, "Deterministic nonperiodic flow", *Journal of Atmospheric Science*, **20**, 130–141 (1963).

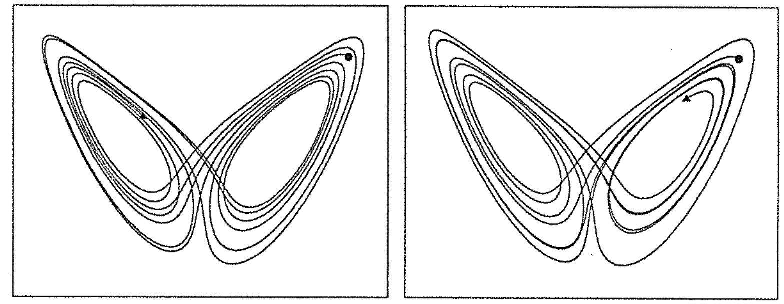

Figure 1.2: Orbits of the Lorenz equations, both with $r = 28$. The initial positions (circle) differ in the fourth significant place, the final positions (triangle) are qualitatively different after a few circuits.

Two typical solutions are shown in figure 1.2, numerically generated by the ODE ORBITS window of CHAOS FOR JAVA.[12] Using Lorenz' choice for the parameters, namely $b = 10$, $\sigma = 8/3$ and $r = 28$, one finds orbits which have become one of the icons of chaos. Each orbit commences from the point displayed as a small circle and ends at the point displayed as a small triangle. The initial position of the two differs only in the fourth significant place of the x-coordinate; it is clear that the final point is on a different wing.

From these static pictures it is not at all evident how the state of the system (which is generated by a continuously travelling point in three-dimensions) moves along its three-dimensional trajectory in time, or even that the orbits are three-dimensional. To see this, you must generate the orbits for yourself using CHAOS FOR JAVA, which will both show you this behaviour and allow you to view the orbits from different view points. You will see that the generating point makes one or more circuits around one of the wings before switching to the other: this process of making circuits then switching continues indefinitely.

The butterfly effect

Lorenz noticed that when he attempted to recompute a given orbit, using the same program on the same computer, he got a different result from the original. This was because his recorded values of x, y and z were less accurate than the internal representation used by the computer, so he was comparing two solutions which differed in their initial state by a small

[12] See appendix A.9 for documentation on the ODE ORBITS window.

amount. The surprising effect is that, after a while, the two solutions don't seem to have much correlation with each other at all.

For example, the two orbits shown in figure 1.2 differ only by a change of initial values of y in the fourth significant place. Even in the short time span (20 units) of the displayed orbits, one sees that they no longer agree except in the most qualitative feature that they are both organised by the same strange attractor. It is not just the growth of error that is involved. After all, the simple linear dynamical model $x_{k+1} = rx_k$ has the property that, if $r > 1$, then an initial difference is magnified by the increasingly large factor r^k as k increases. However, in this linear model the relative error remains at the same level of significance for all k, and the qualitative behaviour of the two solutions is the same in the sense that they look the same over long intervals of time. What we are facing in equations such as Lorenz' is the fact that the relative error quickly becomes as large as the quantities themselves, and that different solutions only have similar qualitative behaviour over relatively short time intervals. That being said, a strange attractor does supply a recognisable structure for the solutions.

This effect, *sensitive dependence* of the evolution of a system to the most infinitesimal changes of initial state is known as the *butterfly effect*, after the title of a talk by Lorenz:[13]

> Predictability: Does the flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?

It encapsulates the question: if Lorenz' equations do not allow long time prediction, why should more complicated dynamical models of the atmosphere do any better? Debate over such questions continues, as does research into weather and climate prediction. An entire chapter of Lorenz' book [16] is devoted to an informed but non-technical discussion of the weather and the implications of chaos for forecasting.

Origin of the Lorenz equations

In the model from which the Lorenz equations are distilled, the focus of interest is on convective fluid motion driven by heating from below, such as might occur locally over warm terrain. Lorenz took a set of seven coupled differential equations (derived by a colleague), ignored four apparently insignificant variables, and investigated solutions of the remaining three coupled equations. This gave him his first real glimpse of infinitely complex behaviour in a simple deterministic system.

Of course the original equations for the fluid motion are infinitely complicated, since they must take account of the temperature and movement at every point in the fluid. Lorenz' equations are the simplest possible reduction which retains at least some interesting and representative behaviour. By using a Fourier representation, they impose a simple dependence of temperature variation on height, whose amplitude is measured by the function $z(t)$. Similarly, the intensity of the resulting convective motion and the horizontal temperature gradient are given fixed functional forms, with amplitudes $x(t)$ and $y(t)$. As for the constants, b is related to the horizontal scale of the convective cells, while σ and r encapsulate some important physical properties of the fluid. The main point is that the equations do arise as an extremely simplified dynamical model of a phenomenon which is important in understanding the atmosphere. More importantly, the infinite complexity is not dependent on having an infinitely complicated system.

[13] The text of the talk is reprinted in Lorenz [16]. He points out that one might equally well ask if the Butterfly can prevent a tornado in Texas.

1.4 Complex behaviour of simple systems

This book is an elementary introduction to the theory of dynamical systems and chaos. The principal aim is to explore the deep relationship between dynamical systems, chaos and fractals, and to uncover structure even where order seems to be absent. We want to understand some of the phenomena which are common across diverse systems, and investigate the mechanisms which make them so. The approach will combine relatively simple mathematics with computer experiments using the program CHAOS FOR JAVA, which has been developed specifically for this purpose.

In the present context, chaos in a dynamical system is a situation where one sees:

(i) Sensitive dependence on the initial conditions, making long-term prediction impossible — the *Butterfly effect.*

(ii) Mixing of the states of the system on ever finer scales so that the trajectories which it may follow become inextricably tangled.

Deterministic dynamical systems may exhibit regular behaviour for some values of their control parameters and irregular behaviour for others. One speaks of *regular* and *chaotic* behaviour in such a system. To quote from an earlier paper of Holmes[14]

> We thus see that deterministic dynamical systems can give rise to motions which are essentially random.

As for the utility of investigating such simple systems, I conclude with another quote from the article by Holton and May

[14] Philip Holmes, "A non-linear oscillator with a strange attractor", *Philosophical Transactions of the Royal Society of London*, **A292**, 419–448 (1979).

The development of dynamical systems theory in general, and Lorenz' contribution and May's review[15] in *Nature* in particular, triggered a change in perception: that a large number of complicated equations were not necessary ... for solutions to be chaotic or turbulent-like. Dynamical systems theory has been driven to the forefront of many fields of science with an impressive number and variety of applications.

Exercises

1.1 Experiment with the ITERATE(1D) window of CHAOS FOR JAVA to find different behaviours for the SINE MAP,

$$x_{k+1} = q \sin \pi x_k, \qquad (0 \leq q \leq 1). \tag{1.2}$$

In particular, find some values of q for which the population reaches a steady value, some where it undergoes periodic cycles, and some where it appears to be disordered.
Pay attention also to the behaviour near to $q = 0.938$, observing what happens just before this regular behaviour sets in, and what happens as it dissolves into disorder again.

[15] Robert M. May, "Simple mathematical models with very complicated dynamics, *Nature*, **261**, 459–469 (1976).

Chapter 2

Orbits of one-dimensional systems

This chapter is an investigation of orbits of discrete one-dimensional dynamical systems, particularly properties of stability and periodicity.

2.1 Discrete dynamical systems

I commence with some definitions. While they may seem rather pedantic at this juncture, I want to emphasise the fact that the concept of orbits is independent of such properties as periodicity or stability.

Definition 2.1 (One-dimensional system) *An equation of the form*

$$x_{k+1} = f(x_k; \mu), \tag{2.1}$$

is called a discrete one-dimensional dynamical system, while the quantity x is called the state variable. The coefficient μ, which is not affected by the iteration, is called a control parameter.

A one-dimensional system has only a single state variable, however some have more than one parameter. The function f must have the property that the domain (input) space is *mapped* to itself, so as to allow for iteration; for this reason I shall refer to functions which determine the behaviour of dynamical systems as *maps* rather than functions. Note that the range (output) space can be either a subset of the domain or the whole of it; the essential point is that the range should not exceed the domain.

If more than one state variable is required to model a system, an equation is required for each variable, and the equations take a multi-dimensional form such as the two-dimensional system

$$x_{k+1} = f(x_k, y_k), \qquad y_{k+1} = g(x_k, y_k).$$

Consideration of such systems, and of continuous systems, is deferred to later chapters.